Copies of the inside front and back covers of the Griffiths text are provided on the last page.

1. The analogue of integration by parts for the divergence is:

$$\int_{\mathcal{V}} f\left(\nabla \cdot \mathbf{A}\right) \mathrm{d}\tau = \oint_{\mathcal{S}} f\mathbf{A} \cdot \mathrm{d}\mathbf{a} - \int_{\mathcal{V}} \mathbf{A} \cdot \left(\nabla f\right) \mathrm{d}\tau.$$

This has the usual integration by parts format in which the derivative is transferred from the $\mathbf{A}$ to f and the product $f\mathbf{A}$ has to be evaluated at the boundaries of the region of integration. In this case, the original integral is over a volume $\mathcal{V}$ and the boundary is the surrounding surface $\mathcal{S}$. Compare to:

$$\int_a^b f\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right) \mathrm{d}x = uv|_a^b - \int_a^b g\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right) \mathrm{d}x.$$

(a) Use integration by parts to evaluate:

$$I = \int_{\mathcal{V}} \left(r^4 - 1\right) \nabla \cdot \left(\frac{\hat{r}}{r^2}\right) \mathrm{d}\tau.$$

For the integration volume, use a sphere of radius R centred at the origin.

(b) Derive the following integration by parts rule for the curl:

$$\int_{\mathcal{S}} f\left(\nabla \times \mathbf{A}\right) \cdot \mathrm{d}\mathbf{a} = \oint_{\mathcal{P}} f\mathbf{A} \cdot \mathrm{d}\boldsymbol{\ell} + \int_{\mathcal{S}} \left(\mathbf{A} \times \nabla f\right) \cdot \mathrm{d}\mathbf{a}.$$

For this derivation, start by using the appropriate product rule for the ∇ operator and then apply Stoke's theorem.

2(a) Starting from:

$$\int_{-a}^{b} f(x) x \frac{\mathrm{d}\delta(x)}{\mathrm{d}x} \, \mathrm{d}x,$$

use integration by parts of show that:

$$x \frac{\mathrm{d}\delta(x)}{\mathrm{d}x} = -\delta(x),$$

where $\delta(x)$ is the one-dimensional Dirac Delta function. Assume that a and b are positive constants such that the point $x = 0$ lies between the integration limits.

(b) Let $\theta(x)$ be the Heaviside step function defined by:

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x <= 0. \end{cases}$$

Starting with:

$$J = \int_{-\infty}^{\infty} f(x) \frac{\mathrm{d}\theta(x)}{\mathrm{d}x} \, \mathrm{d}x,$$

show that $\delta(x) = \frac{\mathrm{d}\theta}{\mathrm{d}x}$ which is one way to define the one-dimensional Dirac Delta function. You'll have to integrate by parts and then use the fundamental theorem of calculus.
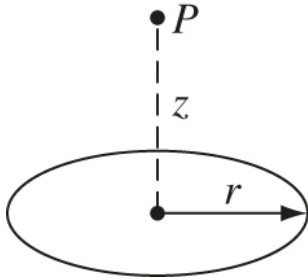
3. Find the electric field a distance z above the centre of a square loop of side length a. Assume that the loop of wire has a uniform linear charge density λ. You may use the fact that the electric field a distance z above the midpoint of a uniformly charge wire of length a is given by:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda a}{z\sqrt{z^2 + \left(\frac{a}{2}\right)^2}} \, \hat{z},$$

(see Griffiths Example 2.2). Does your result make sense in the limit $z \to \infty$?
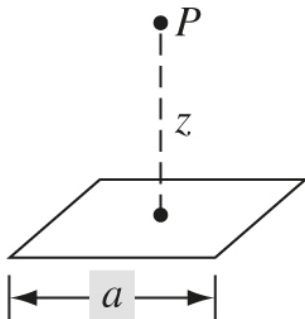
4(a) Find the electric field a distance z above the centre of a uniformly-charged circular loop. The loop has radius r and linear charge density λ. Does your result make sense in the limit $z \to \infty$?

(b) Starting from the result of (a), find the electric field a distance z above the centre of a flat, uniformly-charged circular disk of radius R. The charge per unit area of the disk is σ. Does your result make sense in the limit $z \to \infty$?
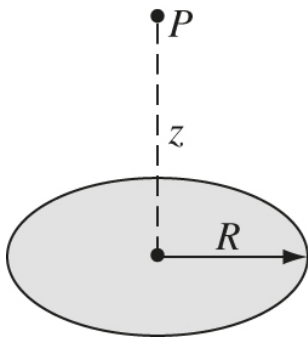
5. A charge q sits at the back corner of a cube, as shown in the figure. What is the electric flux Φ_{E} through the shaded side?
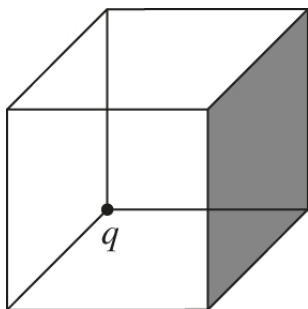
6. Use Gauss's law to find the electric field inside and outside a thin spherical shell of radius R that carries a uniform charge per unit area σ.
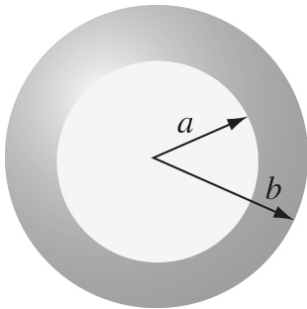
7. A thick spherical shell carries a charge density:

$$\rho(r) = \frac{k}{r^2},$$

for $a < r < b$, where a is the inner radius of the shell and b is its outer radius.

(a) Find the electric field in the following regions of space:

   (i) $r < a$
  (ii) $a < r < b$
 (iii) $r > b$.

(b) Plot the magnitude of $\mathbf{E}$ as a function of r for the case of $b = 2a$.

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient: $\quad \nabla t = \dfrac{\partial t}{\partial x}\,\hat{\mathbf{x}} + \dfrac{\partial t}{\partial y}\,\hat{\mathbf{y}} + \dfrac{\partial t}{\partial z}\,\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z}$

Curl: $\quad \nabla \times \mathbf{v} = \left(\dfrac{\partial v_z}{\partial y} - \dfrac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\dfrac{\partial v_x}{\partial z} - \dfrac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian: $\quad \nabla^2 t = \dfrac{\partial^2 t}{\partial x^2} + \dfrac{\partial^2 t}{\partial y^2} + \dfrac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2 \sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\quad \nabla t = \dfrac{\partial t}{\partial r}\,\hat{\mathbf{r}} + \dfrac{1}{r}\dfrac{\partial t}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \dfrac{1}{r\sin\theta}\dfrac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 v_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}(\sin\theta\,v_\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial v_\phi}{\partial \phi}$

Curl: $\quad \nabla \times \mathbf{v} = \dfrac{1}{r\sin\theta}\left[\dfrac{\partial}{\partial \theta}(\sin\theta\,v_\phi) - \dfrac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$

$$+ \dfrac{1}{r}\left[\dfrac{1}{\sin\theta}\dfrac{\partial v_r}{\partial \phi} - \dfrac{\partial}{\partial r}(rv_\phi)\right]\hat{\boldsymbol{\theta}} + \dfrac{1}{r}\left[\dfrac{\partial}{\partial r}(rv_\theta) - \dfrac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $\quad \nabla^2 t = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial t}{\partial r}\right) + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta \dfrac{\partial t}{\partial \theta}\right) + \dfrac{1}{r^2 \sin^2\theta}\dfrac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $\quad \nabla t = \dfrac{\partial t}{\partial s}\,\hat{\mathbf{s}} + \dfrac{1}{s}\dfrac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \dfrac{\partial t}{\partial z}\,\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \dfrac{1}{s}\dfrac{\partial}{\partial s}(sv_s) + \dfrac{1}{s}\dfrac{\partial v_\phi}{\partial \phi} + \dfrac{\partial v_z}{\partial z}$

Curl: $\quad \nabla \times \mathbf{v} = \left[\dfrac{1}{s}\dfrac{\partial v_z}{\partial \phi} - \dfrac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\dfrac{\partial v_s}{\partial z} - \dfrac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \dfrac{1}{s}\left[\dfrac{\partial}{\partial s}(sv_\phi) - \dfrac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian: $\quad \nabla^2 t = \dfrac{1}{s}\dfrac{\partial}{\partial s}\left(s\dfrac{\partial t}{\partial s}\right) + \dfrac{1}{s^2}\dfrac{\partial^2 t}{\partial \phi^2} + \dfrac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\quad \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\quad \nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\quad \nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\quad \nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\quad \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\quad \nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\quad \nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\quad \nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\quad \nabla \times (\nabla f) = 0$

(11) $\quad \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\quad \int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\quad \int (\nabla \cdot \mathbf{A})\,d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\quad \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho \\[2mm] \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\[2mm] \nabla \cdot \mathbf{B} = 0 \\[2mm] \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\[2mm] \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\[2mm] \nabla \cdot \mathbf{B} = 0 \\[2mm] \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\[2mm] \mathbf{H} = \dfrac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\[2mm] \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \dfrac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy:
$$U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2 \right) d\tau$$

Momentum:
$$\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$$

Poynting vector:
$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

Larmor formula:
$$P = \frac{\mu_0}{6\pi c}q^2 a^2$$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}\,\text{C}^2/\text{Nm}^2$ (permittivity of free space)

$\mu_0 = 4\pi \times 10^{-7}\,\text{N}/\text{A}^2$ (permeability of free space)

$c = 3.00 \times 10^8\,\text{m}/s$ (speed of light)

$e = 1.60 \times 10^{-19}\,\text{C}$ (charge of the electron)

$m = 9.11 \times 10^{-31}\,\text{kg}$ (mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$\begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol\theta} - \sin\phi\,\hat{\boldsymbol\phi} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol\theta} + \cos\phi\,\hat{\boldsymbol\phi} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol\theta} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2 + y^2}/z \right) \\ \phi = \tan^{-1}(y/x) \end{cases}$$

$$\begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol\theta} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol\phi} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases}$$

$$\begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol\phi} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol\phi} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases}$$

$$\begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol\phi} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$